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Weakly non-linear waves in rotating fluids

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The Korteweg–de Vries equation is shown to govern formation of solitary and cnoidal waves in rotating fluids confined in tubes. It is proved that the method must fail when the tube wall is moved to infinity, and the failure is corrected by singular perturbation procedures. The Korteweg–de Vries equation must then give way to an integro-differential equation. Also, critical stationary flows in tubes are considered with regard to Benjamin's vortex breakdown theories.

1. Introduction

This paper considers some aspects of axially symmetric long wave motions of small but finite amplitude in inviscid, incompressible, rotating fluids. The starting point of the investigation is Benjamin (1967*a*), where it is shown that cylindrically symmetric ('support') flows can support stationary waves of finite amplitude under certain circumstances. Pritchard's (1970) experiments confirm the existence of the solitary wave, which is one type predicted by Benjamin.

Here we use Benney's (1966) method to derive equations which describe the later stages in the evolution of these waves. Leibovich (1969) did this for a special class of support flows. Here the general case is considered, and is treated in somewhat more detail. The wave disturbance stream function to lowest order is of the form $\psi = \epsilon\phi(r)A(z,t)$, and, if the motion is confined in a tube, A satisfies the Korteweg–de Vries (KdV) equation,

$$A_t + c_0 A_z = \epsilon c_1 A A_z + k^2 c_2 A_{zzz},$$

as in the problem originally considered by Benney. Here ϵ is a (small) amplitude parameter, k is a (small) 'wave-number' and the c_i are constants. When stationary conditions are assumed, the waves found here coincide with Benjamin's (provided his theory is specialized to conservative flows), as noted in Leibovich (1969).

Partly because of their possible importance to the vortex breakdown phenomenon (Benjamin 1962, 1967*a*), and partly because of their intrinsic interest, we have considered two important examples of critical (in the sense of Benjamin 1962) stationary flows in tubes. From the first example (Poiseuille flow in the axial direction plus solid body rotation), we deduce that the class of support flows, in which the axial velocity vanishes at the wall but the axial vorticity does not, is necessarily subcritical. Therefore, Benjamin's (1962) theory would conclude that a stationary vortex breakdown would be exceptional in a slightly viscous fluid in a tube with rotating walls.

Benjamin's (1962) finite transition theory of vortex breakdown leaves open the important question of the structure of the transition region joining two conjugate flow states. This region is of very great interest (and with possible technological application), particularly since, in some cases at least, it remains fairly well ordered and involves backflows. It is likely that dissipation must be included to properly account for the transition region. On the other hand, it is possible that inviscid solutions may be of value as models. Benjamin (1967*a*) has suggested that a weak breakdown may be represented by a solitary wave. This observation seems worthy of modest amplification (at least), since it provides a model of breakdown which includes the transition region. Our second example is addressed to this, and consists of reasonably detailed numerical computations of the flow pattern due to a solitary wave propagating on a support flow consisting of uniform axial velocity, and Burgers vortex with circulation $\Gamma(r) = K[1-\exp(-\alpha r^2)]$. Harvey (1962) found that this swirl distribution fits the conditions of his vortex experiment, and it seems to be typical of concentrated vortices.

In regard to vortex breakdown, it is interesting to note that some of the *unsteady* features of solutions to the Korteweg–de Vries equation seem to resemble the secondary features of the gentle breakdowns described by Harvey. This is pointed out in §5.

Next, we prove that for a general class of vortices which approach the potential vortex at a distance from the rotation axis, both Benney's (1966) and Benjamin's (1967*a*) perturbation methods break down when the flow field is of large radial extent. This was shown by Benjamin (1967*a*) for the special support flow consisting of a Rankine vortex joined to a potential vortex. This behaviour is not surprising, since the long-wave approximation neglects the wave-number compared to the vorticity, and one might expect the theory to lack uniform validity as the tube wall tends to infinity.

The non-uniformity is corrected by singular perturbation techniques. Conditions in the rotational core are matched to the surrounding (infinite) potential flow. The Korteweg–de Vries equation is now superseded by†

$$A_t + c_0 A_z = \epsilon c_1 A A_z + \frac{k^2 c_3}{2\log(1/k)} \frac{\partial^3}{\partial z^3} \int_{-\infty}^{\infty} \frac{A(\xi, t)\, d\xi}{((z-\xi)^2 + k^2)^{\frac{1}{2}}},$$

and waves can propagate along the vortex core. In the limit $k = 0$, this is the Korteweg–de Vries equation (in the form previously cited).

2. The governing equations

The flow is assumed to have axial symmetry, so the continuity equation in cylindrical (r, θ, z) co-ordinates is automatically satisfied by the introduction of the meridional stream function Ψ, with

$$rw = \Psi_r,$$
$$ru = -\Psi_z,$$

† Pritchard (1970) has independently derived an equivalent version of the stationary form of this equation for the case of a vortex core with a definite sharp boundary. He was motivated by an analysis of a two-layer model of stratified flows considered by Benjamin (1967*b*), where a similar (stationary) equation was found.

where u and w are the radial and axial velocities, respectively, and derivatives are represented by subscripts. The circulation about the axis,

$$\Gamma(r,z,t) = rv(r,z,t),$$

is more convenient to deal with than the azimuthal velocity v. We shall deal with the vorticity equations, and the expressions are simplified if the transformation $y = r^2$ is made. The inviscid form of the equation for azimuthal vorticity may then be written in terms of Ψ, Γ, and y as

$$\mathscr{D}^2\Psi_t + 2\Psi_y\mathscr{D}^2\Psi_z + (2/y)\Gamma\Gamma_z - 2y\Psi_z[y^{-1}\mathscr{D}^2\Psi]_y = 0, \tag{1}$$

where $$\mathscr{D}^2\Psi \equiv 4y\Psi_{yy} + \Psi_{zz}.$$

The equations for radial and axial vorticity are not independent, both being obtained by differentiating the θ-momentum equation, which is

$$\Gamma_t - 2\Psi_z\Gamma_y + 2\Psi_y\Gamma_z = 0. \tag{2}$$

Following the well-established procedure of long wave approximations (Stoker 1957; Benney 1966), introduce separate length scales, say b and L for radial and axial distances, respectively. Velocities may be referred to some typical azimuthal velocity V_0, and time to the convected time L/V_0. Thus, put

$$\begin{aligned} \Psi &= b^2V_0\Psi', \quad \Gamma = bV_0\Gamma', \\ y &= b^2y', \quad t = (L/V_0)t', \\ z &= Lz', \end{aligned}$$

where primed quantities are dimensionless. Defining $k = b/L$, the operator $D^2_{(k)}$ will be defined as $$D^2_{(k)}(\;) = 4y'(\;)_{y'y'} + k^2(\;)_{z'z'} = b^2\mathscr{D}^2.$$

Substituting these expressions into (1) and (2), and dropping the primes, the equations assume the dimensionless form,

$$D^2_{(k)}\Psi_t + 2\Psi_y D^2_{(k)}\Psi_z + \frac{2}{y}\Gamma\Gamma_z - 2y\Psi_z[y^{-1}D^2_{(k)}\Psi]_y = 0, \tag{3}$$

and $$\Gamma_t - 2\Psi_z\Gamma_y + 2\Psi_y\Gamma_z = 0. \tag{4}$$

These equations are exact, within the framework of inviscid fluids, and presently involve only a single parameter k (we have added the subscript (k) to D^2 to recall that the operator depends upon k). In the long wave approximations, k is assumed small, and solutions are sought which hopefully are asymptotically correct as $k \to 0$.

3. Weakly non-linear waves in a tube

Equations (3) and (4) permit the solutions,

$$\Psi = \frac{1}{2}\int_0^y W(y)\,dy,$$

$$\Gamma = \Gamma_s(y),$$

where W and Γ_s are arbitrary functions which represent a cylindrical flow undisturbed by waves. For convenience, the dimensional velocity V_0, to which all velocities have been referred, is taken to be the maximum swirl velocity in this support flow; also, the dimensionless tube radius is taken to be unity. The distribution of Γ and Ψ is assumed to be stable to axisymmetric disturbances, i.e. the stability criterion of Howard & Gupta (1962) is assumed to be satisfied. In our notation, this condition is $\Gamma\Gamma_y \geqslant y^2\Psi_{yy}^2$.

We now seek an axially-symmetric long wave solution of small amplitude measured by the parameter ϵ. Thus, we put

$$\Psi = \tfrac{1}{2}\int_0^y W(y)\,dy + \epsilon\psi(y,z,t),$$
$$\Gamma = \Gamma_s(y) + \epsilon\tilde{\Gamma}(y,z,t).$$

For the case of constant W_0, it has been shown (Leibovich 1969) that an expansion procedure due to Benney (1966) is possible. We shall now give the equations for arbitrary support flows.

To first order in ϵ and k^2, the expansions for ψ and Γ are

$$\psi = \phi_0(y)A(z,t) + \epsilon\phi_1(y)\tfrac{1}{2}A^2 + k^2\phi_2(y)A_{zz} + \ldots,$$
$$\tilde{\Gamma} = \gamma_0(y)A(z,t) + \epsilon\gamma_1(y)\tfrac{1}{2}A^2 + k^2\gamma_2(y)A_{zz} + \ldots,$$

where, to this order, A satisfies the equation,

$$A_t = -c_0A_z + \epsilon c_1AA_z + k^2c_2A_{zzz}, \tag{5}$$

and the constants c_0, c_1 and c_2 are to be determined. Define the operator L to be

$$L(\;) \equiv \frac{d^2}{dy^2}(\;) + q(y)(\;),$$

where
$$q(y) \equiv y^{-1}(W-c_0)^{-2}[y^{-1}\Gamma_s\Gamma_s' - (W-c_0)yW''], \tag{6}$$

and
$$(\;)' \equiv \frac{d}{dy}(\;).$$

Then the functions ϕ_i, $i = 0, 1, 2$ satisfy

$$\left.\begin{aligned} L\phi_0 &= 0,\\ L\phi_1 &= 2\{c_1\phi_0S - \phi_0^2Q\},\\ L\phi_2 &= 2c_2\phi_0S - \tfrac{1}{4}y^{-1}\phi_0,\end{aligned}\right\} \tag{7a, b, c}$$

where
$$\left.\begin{aligned} S(y) &\equiv (W-c_0)^{-1}q(y) + \tfrac{1}{2}(W-c_0)^{-2}W'',\\ Q(y) &\equiv 2(W-c_0)^{-\frac{3}{2}}\frac{d}{dy}[y(W-c_0)^{\frac{1}{2}}q] + (W-c_0)^{-2}(yW')''.\end{aligned}\right\} \tag{8a, b}$$

Furthermore, the corresponding functions γ_i, are given by

$$\left.\begin{aligned} \gamma_0 &= 2\phi_0(W-c_0)^{-1}\Gamma_s',\\ \gamma_1 &= 2(W-c_0)^{-1}\{\phi_1\Gamma_s' - c_1\phi_0(W-c_0)^{-1}\Gamma_s'\\ &\qquad + 2\phi_0^2[(W-c_0)^{-1}\Gamma_s']'\},\\ \gamma_2 &= 2(W-c_0)^{-1}\Gamma_s'[\phi_2 - c_2\phi_0(W-c_0)^{-1}].\end{aligned}\right\} \tag{9a, b, c}$$

The boundary conditions are

$$\phi_n(0) = \phi_n(1) = 0 \quad (n = 0, 1, 2), \tag{10}$$

and arise from the kinematic conditions $u(0) = u(1) = 0$.

The constants c_0, c_1, and c_2 remain to be determined. Fixing $W(y)$, $\Gamma_s(y)$, c_0 is an eigenvalue determined so as to satisfy boundary conditions on ϕ_0. The latter two constants must be chosen so that the right-hand sides of (1*b*) and (7*c*) are orthogonal to ϕ_0, since otherwise the inhomogeneous boundary-value problems (7*b*), (7*c*), and (10) have no solutions. Thus,

$$c_1 = \int_0^1 \phi_0^3 Q\,dy/\Delta, \tag{11}$$

and

$$c_2 = \tfrac{1}{8}\int_0^1 y^{-1}\phi_0^2\,dy/\Delta, \tag{12}$$

where

$$\Delta = \int_0^1 \phi_0^2 S\,dy. \tag{13}$$

Since the integrand in c_1, and that in (13) are divided by $(W-c_0)^2$ and $(W-c_0)^3$, respectively the question of divergent integrals must be faced should $W = c_0$ in the interval (0, 1). However, a theorem of Chandrasekhar's (1961, (78*b*)) ensures that there exist at least two values of c_0, say $c_{(0)}$ and $c^{(0)}$, such that

$$c_{(0)} < \min_{y=(0,1)} W(y),$$

and

$$c^{(0)} > \max_{y=(0,1)} W(y)$$

provided that the flow is stable. Thus, divergent integrals can be avoided.

The eigenvalue problem for ϕ_0 can be approached in a second useful way. Fixing c_0 and Γ, and writing $W(y) = \mu w(y)$, where $w(y)$ is a specified profile function, μ is regarded as the eigenvalue. A similar procedure may be found in Chandrasekhar (1961). This second approach is the one taken in the special examples to be considered.

The wave function A is governed by (5), which is the Korteweg–de Vries equation. It is more conveniently written in terms of the co-ordinate

$$X \equiv z - c_0 t$$

and a slow time $\tau \equiv \epsilon t$ (effects of finite amplitude become important for $\tau = O(1)$), viz.

$$A_\tau = c_1 A A_X + \frac{k^2}{\epsilon} c_2 A_{XXX}.$$

When $\epsilon = O(k^2)$ (there is no loss in taking $\epsilon = k^2$ in such circumstances) the Korteweg–de Vries equation has as permanent wave solutions (Korteweg & de Vries 1895) the solitary wave

$$A = a\,\mathrm{sech}^2\left[\frac{1}{2}\left(\frac{c_1 a}{3c_2}\right)^{\frac{1}{2}}(X + \tfrac{1}{3}ac_1\tau)\right],$$

where a = constant; and cnoidal waves.

Benjamin (1967*a*) has found these same solutions. In fact, if one specializes Benjamin's theory to conservative flows, and the present section to stationary flows, then the two are equivalent.

4. Examples of critical stationary flows with reference to Benjamin's vortex breakdown theory

The concept of a 'critical' stationary flow is central to Benjamin's (1962, 1967*a*) theory of vortex breakdown.

A flow with given $\Gamma_s(y)$ is said to be critical if, with respect to a reference frame in which the axial support velocity is $W(y)$, the 'upstream' speed of the fastest infinitesimal waves vanishes. Since the support flow presents a dispersive medium for infinitesimal waves, with longer waves travelling faster (Benjamin 1962), critical flows refer to infinitesimal waves of extreme length, or $k \to 0$. This condition is, of course, already reflected in the present equations. With regard to long waves of finite amplitude, criticality implies that a finite amplitude wave may remain at rest when viewed in this reference frame. Thus, $c_0 = 0$.

Because of the probable importance of these flows to the phenomenon of vortex breakdown, and because of their inherent interest as wave phenomena, this section is devoted to the study of two important special examples. The first has as support the combination of solid body rotation with Poiseuille flow. Pedley (1969) has shown that this flow is unstable to non-axisymmetric disturbances. Its relevance in practice might therefore be questioned. Nevertheless, it is known to be stable to axisymmetric disturbances, it provides a simple (and important) example of a primary flow with axial shear, and elucidates an entire class of possible flows.

The second example has a support flow with uniform axial velocity and the pseudo-viscous Burgers vortex

$$\Gamma_s = K(1-e^{-\alpha y}). \tag{14}$$

Harvey (1962) found that a circulation distribution of this form agreed well with the flow upstream of observed vortex breakdowns in his experiments. In fact, (14) seems to be typical of a wide class of real concentrated vortex motions.

(i) *Poiseuille flow and solid body rotation*

The support flow is

$$\Gamma_s = y,$$

and

$$W(y) = \mu(1-y).$$

Thus, in order to establish criteria for criticality, we must solve the eigenvalue problem for ϕ_0 (using the second approach mentioned in §3) corresponding to (7*a*).

$$y\left(1-\frac{\mu}{\tilde{c}}y\right)^2 \phi_0'' + \frac{1}{\tilde{c}^2}\phi_0 = 0,$$

$$\phi_0(0) = \phi_0(1) = 0,$$

and

$$\tilde{c} \equiv \mu - c_0.$$

As critical conditions are approached through subcritical ones, $\mu/\tilde{c} \to 1$ from below. On setting $\mu/\tilde{c} = 1-\epsilon_1$, this corresponds to $\epsilon_1 \to 0$ through positive values.

If we put

$$X = (1-\epsilon_1)y,$$

we have†

$$X(1-X)^2\frac{d^2\phi_0}{dX^2}+\frac{1}{(1-\epsilon_1)\tilde{c}^2}\phi_0 = 0, \tag{15}$$

$$\phi_0(0) = \phi_0|_{X=1-\epsilon_1} = 0.$$

The problem for breakdown is (15) with $\epsilon_1 = 0$. Due to the strong singularity at the right end point, the limit problem ($\epsilon_1 = 0$) is not of standard Sturm–Liouville form.

The spectrum involved in this problem is continuous, so that the wave speeds are not uniquely determined from the limit problem‡ To see this, it suffices to consider the structure of the solution near $X = 1$, where the equation may be approximated by

$$(1-X)^2\,\phi_{XX}+\frac{1}{\tilde{c}^2}\phi = 0.$$

If we put

$$\sigma = \left|\frac{1}{4}-\frac{1}{\tilde{c}^2}\right|^{\frac{1}{2}},$$

then the solutions of the approximate equation are in the forms,

$$\phi = (1-X)^{\frac{1}{2}}\{A\sin[\sigma\log(1-X)]+B\cos[\sigma\log(1-X)]\}, \quad \text{for} \quad \tilde{c} < 2,$$

$$\phi = (1-X)^{\frac{1}{2}}\{A+B\log(1-X)\}, \quad \text{for} \quad \tilde{c} = 2,$$

and

$$\phi = (1-X)^{\frac{1}{2}}\{A(1-X)^{\sigma}+B(1-X)^{-\sigma}\}, \quad \text{for} \quad \tilde{c} > 2.$$

Any value of $(\tilde{c})^{-2}$ is a possible eigenvalue. Regardless of the value of $\tilde{c}$, however, the structure of the solutions as revealed by this approximate treatment near $X = 1$ shows that waves of permanent form will not occur, since the constant c_1 in the Korteweg–de Vries equation, as defined by (11) and (13), does not exist. (Neither integral converges, nor does their ratio exist even in a limiting sense.)

The approximate solutions also show that infinitesimal, critical, waves will not exist, since their *energy* will be infinite. It is of interest to note that (Howard & Gupta 1962) all unstable waves have (complex) wave speed, c, between

$$\min W(y) \leqslant |c| \leqslant \max W(y).$$

In this case, $c_0 = \min W(y)$ and hence lies on the boundary of the semicircle.

† This equation has the solution (satisfying $\phi_0(0) = 0$)

$$\phi_0 = X(1-X)^{\beta}F(\beta+1,\,\beta;\,2;\,X),$$

with

$$\beta^2-\beta+1/[(1-\epsilon_1)\tilde{c}^2] = 0,$$

where F is the hypergeometric function. Here $\beta(\epsilon_1)$ must be chosen so that

$$F(\beta+1,\,\beta;\,2;\,1-\epsilon_1) = 0,$$

in order to satisfy the right-hand boundary condition (even in the limit $\epsilon_1 = 0$).

‡ For each $\epsilon_1 > 0$, however, we do have a standard Sturm–Liouville problem and the wave speeds $\tilde{c}(\epsilon_1)$ *are* uniquely determined. It appears that the limiting speed $\tilde{c}(0)$ also exists, and that $\tilde{c} = 2$. Nevertheless, as shown above, waves are not expected in this limit flow.

A similar semicircle theorem holds for the case of stratified flow, where Miles (1961) has shown that isolated neutrally stable waves may exist with speeds inside the semicircle. These waves, however, possess infinite energy. Apparently, the case just considered is an example of the same kind of situation.

The conclusion here, that no meaningful solution exists for critical ($\epsilon_1 = 0$) conditions also applies for $\epsilon_1 < 0$ (supercritical conditions), since the singularity in the equations is simply shifted from the tube wall into the fluid interior. Thus, this kind of support flow is necessarily subcritical.

Furthermore, and more importantly, it is obvious that this analysis applies to any 'viscous' axial velocity profile, where $W(y) \to 0$ as $y \to 1$, whenever the quantity,

$$\Gamma_s \Gamma_s' \neq 0,$$

at the tube wall $y = 1$. In a viscous fluid, this will occur when the tube wall rotates. Thus, we are led to the conclusion that breakdown is not to be expected in a tube if the tube wall rotates, except for the unlikely circumstance in which the vorticity Γ_s' vanishes at the wall.

(ii) *Burgers vortex*

Here the support is $W = \mu =$ constant, and (14).

This model has been considered by Squire (1962) in the case of a radially unbounded flow field.

By our convention, the maximum value of $V(r) = \Gamma_s/r = 1$, and, since the maximum occurs at $\alpha y = 1{\cdot}2565$, $K = k_0^2 \alpha^{-\frac{1}{2}}$, where $k_0^2 = 2{\cdot}4698$ and is independent of α. Thus, the swirl depends upon the single parameter α (which determines the position y, at which $V(r)$ is maximum).

The problem for ϕ_0 is, therefore,

$$\left.\begin{aligned} \phi_0'' + \left[\frac{k_0^2}{y\mu}\right]^2 e^{-\alpha y}(1 - e^{-\alpha y})\phi_0 = 0, \\ \phi_0(0) = \phi_0(1) = 0. \end{aligned}\right\} \qquad (16)$$

The most useful single piece of information, that can be given concerning a vortex breakdown in a particular experiment, is probably a swirl angle parameter θ_c, at which breakdown is observed. According to Benjamin's (1962) theory of vortex breakdown, this should correspond to a nearly critical support flow. For a support with constant axial velocity μ, the angle may be defined as $\theta_c = \tan^{-1}(V_m/\mu)$, when the flow is critical. Because of the scaling used here V_m, which is defined as the maximum value of the support swirl velocity, is unity. From Sturm–Liouville theory, there are a countably infinite number of eigenvalues $1/\mu^2$ for the eigenvalue problem for ϕ_0, and these may be ordered such that

$$\mu_1^2 > \mu_2^2 > \dots \mu_k^2 > \dots.$$

The largest value of μ corresponds to critical flow in the sense of Benjamin and should be used to calculate θ_c.

We have found the largest wave speeds $\mu(\alpha)$ for values of α ranging from 2 to 16 steps of 2 by numerically solving the eigenvalue problem (16) for the first eigenfunction using a Runge–Kutta procedure and a shooting technique. The results for $\tan\theta_c$ as a function of the parameter α are shown in the table. We note that for $\alpha = 14$, which is Harvey's (nominal) value, $\theta_c \doteqdot 46°$. This compares to the measured value of θ_c of about $49° \pm 2{\cdot}2°$.

α	2	4	6	8	10	12	14	16
$\tan\theta_c$	1·260	1·124	1·096	1·057	1·044	1·035	1·029	1·025

For the larger values of α, $\tan\theta_c$ does not differ much from Squire's calculation, where $\phi_0' \to 0$ as $y \to \infty$. He found that $\tan\theta_c = 1$.

We have also calculated the eigenfunction $\phi_0(r)$, streamlines, and velocities for the solitary wave for the case $\alpha = 14$. The eigenfunction is plotted in figure 1, where we have used a normalization for ϕ_0 such that $d\phi_0/dy = 1$ at the axis $y = 0$.

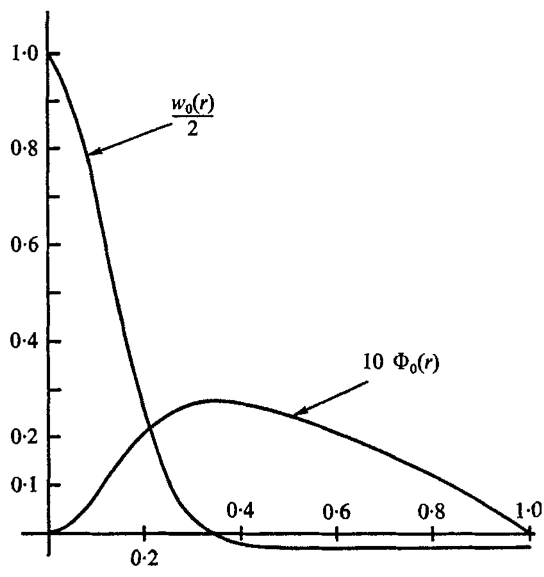

FIGURE 1. The eigenfunction $\phi_0(r)$, and the curve $\frac{1}{2}w_0(r)$. The quantity $-\tilde{\epsilon}w_0(r)$ represents the perturbation axial velocity at $z = 0$. The example here is for the support flow $\Gamma_s(y) = 1 - e^{-14y}$, corresponding to Harvey's experiment.

The parameters occurring in the equation for $A(z,\tau)$ were found to be $c_1 = -0{\cdot}7506$ and $c_2 = 0{\cdot}0158$. For a real solitary wave, the 'amplitude', a, of the solitary wave must be negative, since c_1 is negative. In fact, if we put $\tilde{\epsilon} = -\epsilon a$, $Z = z\epsilon^{-\frac{1}{2}}$, then it is seen that the solitary wave solution depends upon the single (positive) parameter $\tilde{\epsilon}$, which represents one-half of the maximum axial velocity disturbance, and the stream function assumes the form,

$$\Psi = \tfrac{1}{2}Wr^2 - \tilde{\epsilon}\phi_2(r)\,\mathrm{sech}^2\left[\frac{1}{2}\left(\frac{\tilde{\epsilon}|c_1|}{3c_2}\right)^{\frac{1}{2}} Z\right].$$

In this expression, the length scales for r and Z are the same as both are referred to the tube radius b. Since the solitary wave progresses in the negative z-direction at a speed $\tilde{\epsilon}|c_1|/3$ with respect to the co-ordinate system moving with speed c_0, a

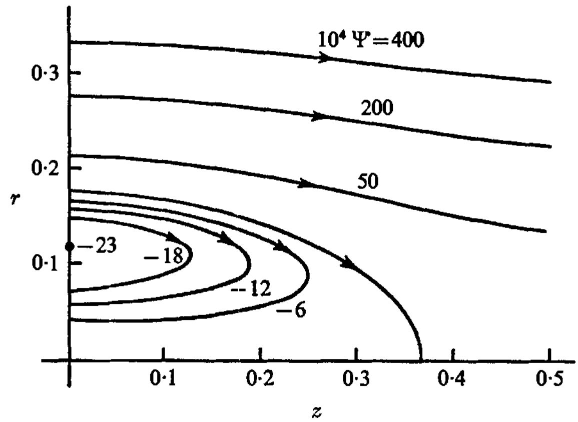

FIGURE 2. Streamlines corresponding to figure 1 in the region of a 'breakdown' eddy, with $\tilde{\epsilon} = 1{\cdot}0$. Unit of length for both axes is the tube radius.

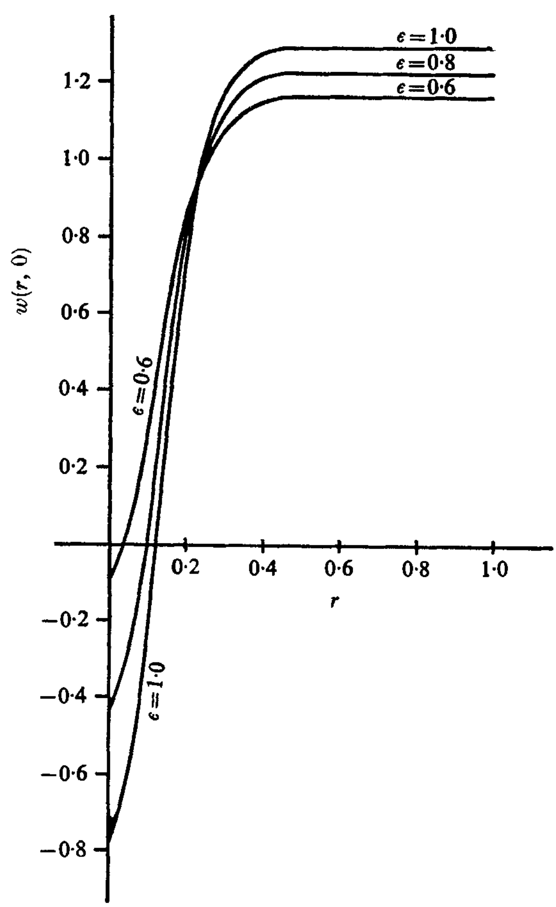

FIGURE 3. The total axial velocity in the breakdown at the plane of symmetry $Z = 0$, for various $\tilde{\epsilon}$, corresponding to figures 1 and 2.

consistent first-order stationary theory should require the wave speed to vanish to first order, or $c_0 - \frac{1}{3}\tilde{\epsilon}|c_1| = 0$. The required stream speed W is, therefore, slightly supercritical, with

$$W = \mu + \tfrac{1}{3}\tilde{\epsilon}|c_1|.$$

Streamlines for $\tilde{\epsilon} = 1{\cdot}0$, and velocities for $\tilde{\epsilon} = 0{\cdot}8$ and $1{\cdot}0$ are shown in figures 2–4. Despite the fact that the rationally established limits of applicability of the solution are violated if $\tilde{\epsilon}$ is not small, the value of $\tilde{\epsilon} = 1{\cdot}0$ seems to compare favourably in many respects with Harvey's observations. It closely approximates both the length and maximum radial extent of the eddy, and faithfully reproduces its shape. One might conjecture that similar wave mechanisms are possible for $\tilde{\epsilon}$ not small.

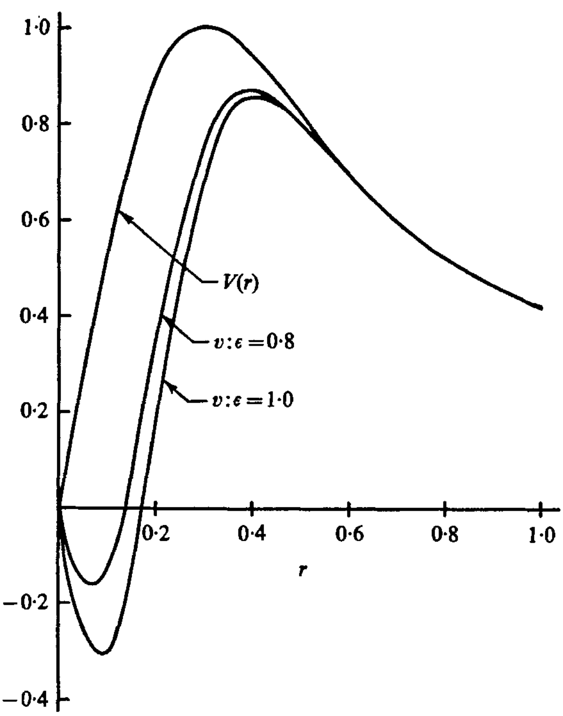

FIGURE 4. Base flow aximuthal velocity and the total azimuthal velocity, corresponding to figure 3.

5. Similarities between unsteady features of Korteweg–de Vries and observations of vortex breakdown

In some of Harvey's photographs, a second breakdown, definitely smaller than the main breakdown and downstream from it, was observed. Generally, the last breakdown was followed by a region of smaller amplitude unsteady flow.

Benjamin (1962, 1967*a*) has supplied an explanation for these secondary features when they occur. As the author understands it, in Benjamin's view of the phenomenon, a 'finite transition' (describable as a large amplitude wave of unknown structure?) can switch on a wave-train, perhaps resembling the cnoidal waves. The main breakdown is then the first wave in the train, and the secondary breakdowns and unsteadiness are the vestiges of the train, which presumably cannot be maintained. The mathematical basis for Benjamin's suggestions are to be found in his 1967*a* paper, which established the possibility of weakly non-linear stationary waves.

This interpretation, which may be correct, is not the only one that may be inferred from the consideration of weakly non-linear waves. An alternative approach to the secondary features in the wake of the main breakdown is to identify them with unsteady effects arising from the KdV equation (cf. Gardner, Greene, Kruskal & Miura, 1967; Zabusky 1968). For $t < \infty$, more than one 'solitary' wave can co-exist, and for large but finite time, the sequence of waves will be ordered according to amplitude (and hence speed), with the largest in the lead. An oscillating, dispersing 'tail' with a maximum amplitude smaller than the smallest solitary wave develops outwards in a direction opposite to that of the solitary wave motion.

Thus, one may advance the hypothesis that the main and secondary breakdowns are co-existing solitary waves with a tendency to slowly part company, and that the unsteadiness downstream of the last breakdown is the small amplitude KdV 'tail'.

An explanation like this is open to the criticism of placing excessive faith in a conservative model when dissipation is likely to be very important in the real event. It is known (e.g. Mei 1966) that adding a certain kind of dissipative term to the KdV equation can drastically alter the character of the solutions. However, in the present case, the inclusion of weak viscous effects (ignoring wall boundary layers) can be accomplished using the procedure of Ott & Sudan (1970), and does not change the nature of solitary waves. This does not obviate the criticism, but it would seem to reduce its force.

6. Failure of the method for radially unbounded flow

When the flow field is radially infinite, the appropriate boundary condition on ϕ_0 was given by Squire (1962), i.e.

$$\frac{d\phi_0}{dy} \rightarrow 0,$$

as y (or r) $\rightarrow \infty$. For the perturbation procedure that leads to the KdV equation to remain valid, the coefficients c_1 and c_2 must exist. In particular, since c_1 presents no difficulties, it is only the integral,

$$c_2 = \tfrac{1}{8}\int_0^{(b/r_0)^2} y^{-1}\phi_0^2\,dy,$$

whose convergence as $b/r_0 \rightarrow \infty$ is in doubt. The upper limit represents the tube wall, if we take the natural course of changing the unit of length from the dimensional tube radius b to r_0, the radius at which the swirl velocity is maximum.

Should the coefficients c_1 and c_2 remain finite, there would be no reason to reject the solutions found here. This is true even though all 'reasonable' vortex cores in fluids of great extent are embedded in potential flows, where axial and radial gradients must be comparable, since in the potential region negligibly small disturbances would be predicted.

The function ϕ_0 can satisfy the boundary condition at infinity if it tends to any constant, including zero. Furthermore, if it tends to zero, the integral for c_2 converges. Benjamin (1967*a*), however, showed that $\phi_0 \to$ non-zero constant and $c_1 \to \infty$ logarithmically as $b \to \infty$, for the special case of the combined vortex (in which $\Gamma_s = Ky$, for $y < y_0$, and $\Gamma_s = Ky_0$, for $y > y_0$). Here we show that his conclusion remains in force for any Γ_s that tends to a constant (potential flow) fast enough. It is assumed that $W(y) \to W_\infty$ (a constant) at least as fast as Γ_s.

Under these conditions, the function $q(y) \to 0$ in the equation,

$$\phi_0'' + q(y)\phi_0 = 0,$$

as $y \to \infty$.

That this equation may have no solutions with the property $\phi_0 \to 0$ and $y \to \infty$, when $q \to 0$, is made plausible by consideration of the two simple examples,

$$\left.\begin{aligned} q &= \lambda^2 y^{-\alpha}, \quad \alpha > 0, \\ q &= \lambda^2 e^{-\beta y}, \quad \beta > 0, \end{aligned}\right\} \tag{17a,b}$$

both of which may be solved explicitly. The asymptotic results for (17*a*) are as follows:

$$\phi_0 \sim y^{\alpha/4}\{A \cos[\lambda y^{1-\alpha/2} + \Omega] + B \sin[y^{1-\alpha/2} + \Omega], \quad \text{if} \quad \alpha < 2,$$

(where Ω is a constant);

$$\phi_0 \sim y^{\frac{1}{2}}\{Ay^{\frac{1}{2}\sqrt{(1-\lambda)}} + By^{-\frac{1}{2}\sqrt{(1-\lambda)}}\}, \quad \text{if} \quad \alpha = 2,$$

where A and B are complex if $\lambda > 1$, and,

$$\phi_0 \sim A + By\left\{1 + \sum_{m=1}^{\infty} c_m y^{(2-\alpha)m}\right\},$$

if $\alpha > 2$, but $a \neq 2 + (1/n)$, $n = 1, 2, \ldots$ with c_r known constants depending on λ while in the last case, $\alpha = 2 + (1/n)$, $n =$ integral, one must add to the foregoing a term $B\tilde{c} \log y$ where $\tilde{c}$ is a known constant. For (17*b*), one has

$$\phi_0 \sim A + By,$$

which is obtained as well in the case of (17*a*) for $\alpha > 3$.

For the more general case of unspecified (positive) $q(y)$, it is sufficient that

$$\int_Y^\infty y^2 q(y)\, dy < \infty, \tag{18}$$

for some number Y, in which case the asymptotic form of ϕ_0 is

$$\phi_0 \sim A + By.$$

This is seen by writing the original differential equation as a Volterra equation, and adapting standard methods (Jeffreys 1962). The Volterra equation is

$$\phi_0(y) = A + By + \int_y^\infty (y - \eta)\, q(\eta)\, \phi_0(\eta)\, d\eta.$$

We construct two independent solutions of the differential equation by considering in turn the two integral equations,

$$\left.\begin{aligned} u_1 &= 1+\int_y^\infty (y-\eta)\,q(\eta)\,u_1(\eta)\,d\eta \\ \text{and}\qquad u_2 &= y+\int_y^\infty (y-\eta)\,q(\eta)\,u_2(\eta)\,d\eta. \end{aligned}\right\} \tag{19a,b}$$

For (19a) consider the sequence of functions $f_n(y)$:

$$f_1 = 1,$$

$$f_n = \int_y^\infty (y-\eta)\,q(\eta) f_{n-1}(\eta)\,d\eta.$$

Then, for $n > 1$, $\quad |f_n(y)| \leqslant \max\limits_{(Y,\infty)} |f_{n-1}(y)|\,\delta(y) \leqslant [\delta(y)]^{n-1},$

where $$\delta(y) = \int_y^\infty \eta q(\eta)\,d\eta$$

is by hypothesis $o(y^{-2})$. Thus, the series $f = \sum\limits_{n=1}^{\infty} f_n(y)$ is absolutely and uniformly convergent in the interval $y > Y$. In particular,

$$|u_1-1| \leqslant \frac{\delta(y)}{1-\delta(y)}, \quad \text{for} \quad y > Y_1,$$

where Y_1 is such that $\delta(Y_1) \leqslant \frac{1}{2}$. Then $u_1 = 1$, with an error uniformly of $O[2\delta(y)]$ as $y \to \infty$.

A similar procedure can be used to construct a sequence g_n, with $g_0 = y$, to construct a uniformly and absolutely convergent series,

$$u_2 = y + \sum_{n=1}^{\infty} g_n(y),$$

for $y > Y$, where Y is given by the hypothesis (18). u_2 is a solution of the differential equation (5), with the uniformly valid asymptotic estimate

$$u_2 - y - \sum_{n=1}^{N} g_n = O\{[\delta(y)]^N\}.$$

Since $u_1 \sim 1$, $u_2 \sim y$, the Wronskian of u_1, u_2 is non-zero, and hence the two solutions are independent. The general solution is a linear combination of u_1, u_2, so that

$$\phi_0 = Au_1 + Bu_2 \sim A + By, \quad \text{as} \quad y \to \infty. \tag{20}$$

Therefore, ϕ_0 can vanish only if $\phi_0 \equiv 0$. This completes the proof.

The same techniques can no doubt be applied to the case where $q \sim y^{-\alpha} + \lambda(y)$, where $\lambda(y) = o(y^{-\alpha})$ and $\alpha > 0$. Then it is expected that the solutions discussed in case (a) above are leading asymptotic estimates of the solution, but we have not checked the details.

Thus, the perturbation method systematically employed here, in which the wave-number is assumed to be $O(\epsilon^{\frac{1}{2}})$ and, therefore, is neglected compared to $q(y)$, leads to a non-uniform behaviour in the limit $b \to \infty$.

7. Radially unbounded flow

It is now supposed that the flow field is unbounded, and that, except for a core region surrounding the axis, the support motion is irrotational, with a constant, non-zero circulation about the z-axis. Near the rotation axis, the support flow is rotational, and the maximum value of the undisturbed swirl velocity, say V_0, is attained at the (dimensional) radius $\bar{b}$. The circulation is assumed to approach the constant value in the potential region exponentially in radius. Burgers vortex (14) is the prototype of the behaviour assumed. According to the last section, we expect that a singular perturbation is required.

As before, we take L, a length comparable to the wavelength, as the scale for axial variations. Two length scales are now required for radial distance. Clearly, $\bar{b}$ is an appropriate scale for radius in the core, so we define an 'inner' co-ordinate,

$$r^* = \bar{b}r.$$

Here r^* is the dimensional radius. Wave propagation, which is made possible by the vorticity in the core, is sought. A wave propagating along the core appears to the potential flow something like a slender body moving on the axis. In the potential flow, radial and axial gradients are comparable, so an 'outer' co-ordinate ρ is defined by

$$r^* = L\rho,$$

thus,

$$\rho = kr,$$

where $k = \bar{b}/L$ and is supposed small, as before.

In the outer region, the deviation of the support flow from potential flow is exponentially small. Hence, to all orders in ϵ, the equation for the stream function is

$$\frac{\partial^2\Psi}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\Psi}{\partial\rho} + \frac{\partial^2\Psi}{\partial z^2} = 0. \tag{21}$$

The solution to (21) which approaches the uniform stream at large distances is

$$\Psi = \tfrac{1}{2}W_\infty\rho^2 + \epsilon\psi; \quad \psi = \frac{\rho}{2\pi}\int_{-\infty}^{\infty} \zeta F(\zeta, t)\, K_1(\rho|\zeta|)\, e^{i\zeta z} d\zeta.$$

There is no disturbance to the swirl velocity.

We assume that the function $F(\zeta, t)$ in the Fourier transform of the disturbance ψ, has an asymptotic approximation for small ϵ of the form,

$$F = \sum_{i=1}^{\infty} f_i(\epsilon, k) F_i(\zeta, t),$$

where the $f_i\,(\epsilon, k)$ form an asymptotic sequence.

Then the Fourier transform

$$\bar{\psi} = \int_{-\infty}^{\infty} \psi(\rho, z, t)\, e^{-i\zeta z} dz$$

is, asymptotically

$$\bar{\psi} = \left[\sum_{i=1}^{\infty} f_i(\epsilon, k)\, F_i(\zeta, t)\right] \rho\zeta K_1(\rho|\zeta|).$$

It is anticipated that a theory valid to $O(\epsilon^2)$ and $O(\epsilon k^2)$ will be sufficient to describe the formation of finite amplitude waves, since this proved sufficient in the case of waves in a tube.

In order to accomplish the matching with the vortical core, we rewrite $\bar{\psi}$ in terms of the inner co-ordinate $r = \rho/k$, and expand the result for $k \to 0$, r fixed in the usual way, with a view towards applying the asymptotic matching principle (van Dyke 1964; Fraenkel 1969):

$$\bar{\psi} = F(\zeta, t)\,\mathrm{sgn}\,(\zeta)\,[1 + \tfrac{1}{2}k^2r^2\zeta^2[\log k + \log r + \log(\tfrac{1}{2}|\zeta|) + \gamma - \tfrac{1}{2}]] + O(k^4 \log kF), \quad (22)$$

where γ is Euler's constant. (γ is reserved for Euler's constant in the remainder of the paper.)

As in the case of flow in a tube, the inner stream function is

$$\Psi = \int_0^r rW(r)\,dr + \epsilon\chi,$$

where $W(r)$ is the (specified) support axial velocity distribution and $\lim_{r\to\infty} W = W_\infty$, and where it is assumed that the disturbance can be written in the form (again following Benney 1966),

$$\chi = \phi_0(r)A(z,t) + \sum_{n=1}^{\infty} g_n(\epsilon, k^2)\,\phi_n(r)\mathscr{C}_n(A),$$

where the $\mathscr{C}_n(A)$ are operators on the wave function A. The gauge functions g_n need not form an asymptotic sequence, and in particular two or more of the g_n may coincide.

With the transformations, $X = z - c_0t$, $\tau = \epsilon t$ used earlier, the equation for A is

$$A_\tau = \frac{1}{\epsilon}\frac{\partial}{\partial X}\left\{\sum_{n=1}^{\infty} g_n c_n \mathscr{C}_n(A)\right\}.$$

As before, the constants c_n are chosen so that the problems for the $\phi_n(r)$ are solvable. The identification of $g_1 = \epsilon$, $g_2 = k^2$, $\mathscr{C}_1 = \frac{1}{2}A^2$, $\mathscr{C}_2 = A_{XX}$ is required as before to permit separability of the $O(\epsilon^2)$ and $O(\epsilon k^2)$ equations. Furthermore, ϕ_0, ϕ_1, and ϕ_2 are governed by ($7a, b, c$) as in the case of flow in tubes.

Up to this point, the development is identical to that of §3. However, to match χ with its potential counterpart ψ to $O(k^2)$, it is necessary to add two additional terms to the inner expansion, one of order $k^2 \log k$, and another of order k^2. Therefore, we take

$$g_3 = k^2 \log k, \quad g_4 = k^2,\dagger$$

and then the functions $\phi_{3,4}$ satisfy the equations

$$L\phi_{3,4} = 2c_{3,4}\phi_0 S(y). \quad (23)$$

† Since $g_2 = g_4$, one sees that $\phi_2 A_{XX}$ is a particular solution for the $O(k^2)$ equations, while $\phi_4\mathscr{C}_4(A)$ is a homogeneous solution.

The boundary condition at $r = 0$ on all ϕ_n is

$$\phi_n(0) = 0, \tag{24}$$

and conditions as $r \to \infty$ are determined from the matching procedure.

The asymptotic matching principle (van Dyke 1964; Fraenkel 1969) is used to connect the inner and outer expressions for the Fourier transforms of the disturbance streamfunction. As usual, those terms in the sequence which are $O(k^2 \log k)$ and $O(k^2)$ must be considered together to effect a match. One needs the asymptotic behaviour of the ϕ_n, which depend on the condition (18) of §6, and on the inhomogeneous terms in the governing equations. For $n = 0, 1, 3, 4$ we have (20), or

$$\left.\begin{aligned} \phi_n &= a_n + b_n r^2, \\ \text{while} \quad \phi_2 &= a_2 + b_2 r^2 - \tfrac{1}{4} b_0 r^4 - \tfrac{1}{4} a_0 r^2 \log r^2, \end{aligned}\right\} \tag{25}$$

where the a_n, b_n are constants.

The match of 1 term inner and outer expansions provides (the overbar indicates a Fourier transform)

$$f_1 = 1, \quad F_1 = a_0 \bar{A}(\zeta, t) \operatorname{sgn} \zeta, \quad b_0 = 0,$$

so that the correct boundary condition on ϕ_0 is

$$\phi_0 \to a_0 \equiv \phi_\infty \,(\text{say}),$$

as $r \to \infty$.

The match of 3 term inner and outer expansions provides the forms for g_3 and g_4 already anticipated, and, in addition

$$f_2 = \epsilon, \quad \overline{\mathscr{C}}_3 = \zeta^2 \bar{A}, \quad \overline{\mathscr{C}}_4 = \zeta^2 \bar{A} \log(\tfrac{1}{2}|\zeta|),$$
$$b_1 = 0, \quad b_2 = \tfrac{1}{2}\phi_\infty(\tfrac{1}{2} - \gamma), \quad b_3 = b_4 = \tfrac{1}{2}\phi_\infty.$$

At this point, the a_n ($n = 1, \ldots, 4$), remain arbitrary, as does $F_1(\zeta)$. These quantities will (presumably) be fixed at the next stage of matching.

The constant ϕ_∞, which is also arbitrary, is fixed by initial conditions in the wave.

Solvability requirements for the ϕ_n, which must now satisfy the boundary conditions:

$$\begin{aligned} &\phi_n(0) = 0, \quad \text{all} \quad n, \\ &\phi_0 \to \phi_\infty, \\ &d\phi_1/dy \to 0, \\ &d\phi_2/dy \to \tfrac{1}{2}\phi_\infty(\tfrac{1}{2} - \gamma), \\ &d\phi_{3,4}/dy \to \tfrac{1}{2}\phi_\infty, \quad \text{as} \quad y \to \infty, \end{aligned}$$

determine the values of the constants c_n appearing in the equation for A. They are found to be (in the appendix):

$$\left.\begin{aligned} c_1 &= \int_0^\infty \phi_0^3 Q(y)\, dy / \Delta_\infty, \\ c_2 &= -\tfrac{1}{4}\gamma \phi_\infty^2 / \Delta_\infty, \\ c_3 &= c_4 = \tfrac{1}{4}\phi_\infty^2 / \Delta_\infty, \\ \Delta_\infty &= \int_0^\infty \phi_0^2 S(y)\, dy. \end{aligned}\right\} \tag{26a–d}$$

The expression for c_1 is the same as would be found for the case of a tube, if the tube radius were formally extended to infinity.

The Fourier transform of the 'wave' equation for A is

$$\bar{A}_\tau = i\zeta\{\tfrac{1}{2}c_1\bar{A}^2 + c_3(k^2/\epsilon)\,\zeta^2\bar{A}[\gamma + \log(\tfrac{1}{2}k|\zeta|)]\}, \tag{27}$$

with an error of order $(k^4/\epsilon)\log k$. Thus, we now expect that the length scale that will produce waves of permanent form is given by the relation $k^2 \log 1/k = O(\epsilon)$ instead of $k = O(\sqrt{\epsilon})$ as in the case of finite flow fields. Thus, we shall formally put

$$\epsilon = k^2 \log(1/k)$$

in the wave equation. With this choice, as $k \to 0$, (27) formally reduces to the (transform of the) KdV equation. However, the reciprocal logarithm error is large. For the equation to have practical utility, one must retain the $(\log k)^{-1}$ term. Since

$$\gamma + \log(\tfrac{1}{2}k|\zeta|) = -K_0(k|\zeta|)$$

with an error $O(k^2 \log k)$, the dispersive term suggests the possibility that if higher order matchings were carried out, the modified Bessel function would emerge. At any rate, there is no loss in accuracy in making the replacement to the present order, and we shall do so. Upon inversion then, the equation for A in real space is

$$A_\tau = c_1 A A_X + \frac{c_3}{2\log 1/k}\frac{\partial^3}{\partial X^3}\int_{-\infty}^{\infty}\frac{A(\xi,\tau)\,d\xi}{[(X-\xi)^2+k^2]^{\frac{1}{2}}}. \tag{28}$$

The form of the integral is interesting. If A were independent of the parameter k, the integral would represent the potential in the (X, k) plane due to a line source distribution of strength A placed upon the $k = 0$ axis. As an alternative to the Fourier transform demonstration, use can be made of the known behaviour of potentials for slender body theory to show that KdV is approached as $k \to 0$ from this equation directly.

Presumably, for k small enough, we should expect that (28) has stationary solitary wave solutions, although this is a difficult matter to prove. A computer study of equation (28) is in progress.

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Appendix

To find the constants c_n, consider first the homogeneous equation

$$L\phi = \phi'' + q(y)\phi = 0.$$

This is satisfied by ϕ_0, and we suppose that a solution for ϕ_0 has been found. Then we have $\phi_0(0) = 0$, $\phi_0(\infty) = \phi_\infty$. Given ϕ_0, a second solution is

$$\theta = \phi_0(y)\int_{\nu}^{y}\frac{d\eta}{\phi_0^2(\eta)},$$

where ν is an arbitrary positive constant. If the hypothesis (18) of § 6 is satisfied, then

$$\theta = \frac{1}{\phi_\infty} y + o(1)$$

as $y \to \infty$. The Wronskian of the two solutions, θ and ϕ_0, is unity.

As $y \to 0$,

$$\theta = -\frac{1}{\phi_0'(0)} + \frac{\phi_0(y)}{[\phi_0'(0)]^2} \int_\beta^y q(\eta)\, d\eta + O(y).$$

If the support flow vorticity on the axis does not vanish, and for real vortices it does not, then for some constant λ, $q(y) = \lambda/y + O(1)$ as $y \to 0$. Thus,

$$\theta = \frac{1}{\phi_0'(0)}[-1 + \lambda y \log y] + O(y),$$

as $y \to 0$.

The general solution of the inhomogeneous equation,

$$L\tilde{\theta} = R(y),$$

is

$$\tilde{\theta} = a\phi_0 + b\theta + \int^y G(y,\eta) R(\eta)\, d\eta,$$

where

$$G(y,\eta) \equiv \theta(y)\phi_0(\eta) - \phi_0(y)\theta(\eta).$$

With these preliminaries, let us begin with the problems for ϕ_3 and ϕ_4. Since they are identical, $\phi_3 = \phi_4$, $c_3 = c_4$, and the solution which satisfies the boundary condition at infinity is

$$\phi_3 = \left\{a_3 + 2c_3 \int_y^\infty \phi_0(\eta)\theta(\eta) S(\eta)\, d\eta\right\} \phi_0(y) + \left\{\tfrac{1}{2}\phi_\infty^2 - 2c_3 \int_y^\infty \phi_0^2(\eta) S(\eta)\, d\eta\right\} \theta(y).$$

The integrals in this expression converge for all $y \geqslant 0$, by the hypothesis (18) on $q(y)$ (to which $S(y)$ is directly related). As $y \to 0$, the boundary condition $\phi_3(0) = 0$ can be met only if the coefficient of $\theta(y)$ vanishes. Hence c_3 and c_4 must have the values (22 c).

Turning to the ϕ_2 problem, we have as general solution

$$\phi_2 = a_2\phi_0(y) + b\theta(y) - \tfrac{1}{4}\int_0^y G(y,\eta)\frac{\phi_0(\eta)}{\eta}\, d\eta - 2c_2 \int_y^\infty G\phi_0 S\, d\eta.$$

As $y \to \infty$, the first integral is asymptotic to

$$\tfrac{1}{4} y\phi_\infty(1 - \log y) + O(1),$$

so that in order to satisfy the boundary condition as $y \to \infty$ we must select $b = -\frac{1}{2}\gamma\phi_\infty^2$. Thus, to eliminate the $\theta(y)$ term as $y \to 0$, as required by the boundary condition there, we must select $c_2 = -\frac{1}{4}\gamma\phi_\infty^2/\Delta_\infty$.

For ϕ_1, we may write the solution as

$$\phi_1 = a_1\phi_0 + 2\int_0^y G(y,\eta)\{c_1\phi_0(\eta)S(\eta) - Q(\eta)\phi_0^2(\eta)\}\, d\eta.$$

The condition on the axis is automatically satisfied, and as $y \to \infty$,

$$\phi_1 \sim \frac{1}{\phi_\infty} y \left\{ c_1 \Delta_\infty - \int_0^\infty Q(\eta)\, \phi_0^2(\eta)\, d\eta \right\} + O(1).$$

Since ϕ_1 is $O(1)$ in this limit, the value (26a) must be chosen for c_1.

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